MODERN DECISION THEORY

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I. Two person zero sum games

The publication of the theory of games (von Neumann and Morgenstern 1944) is a landmark in modern economics. Its influence in the field of mathematical economics has been profound. But it is also fundamental in the applied field of operations research, which is nothing else but the econometrics of the enterprise, both private and public.

We will consider here just two person zero sum games. This is perhaps the most valuable part of modern strategic game theory, and certainly the only case in which a complete and satisfactory solution is available.

Consider the case in which A plays against B. A strategy is the totality of all possible moves of a player. Consider the case where the strategies of A are A_1 and A_2. Similarly the strategies of B are B_1, B_2 and B_3. Then we might construct a gain matrix for A which is at the same time the loss matrix of B, since what A gains B loses and *viceversa*.

TABLE 1

Gain of A=Loss of B

Strategy of B

Strategy of	A	B_1	B_2	B_3	min. of row
	A_1	21	11*	31	11*
	A_2	32	0	4	0
Maximum of column		32	11*	31	

Consider now the situation of the player *A*. If he uses his strategy A_1 he will gain (*B* will lose) 21, 11 or 31 money units according to the strategy *B* uses. But *B* is an intelligent opponent. What *A* gains *B* loses. Hence if *A* uses his strategy A_1 he can only count on the minimum of the first row, 11. If *A* uses his first strategy he must count on *B* using his strategy B_2 which minimises his gain, which is at the same time the loss of *B*.

Suppose *A* uses his strategy A_2. Then he will gain (*B* will lose) 32, 0 or 4 money units depending on which strategy *B* utilizes. But it is in the interest of *B* to minimise A's gains which are B's losses. Hence *A* can again only count on the row minimum 0, which corresponds to B's strategy B_2.

Now consider the situation of *B*. Table 1 presents A's gains which are at the same time B's losses. If *B* utilizes his strategy B_1 he will lose 21 or 32 money units depending on the strategy which *A* utilizes. But since B's losses are A's gains he can only count on the maximum of the first column, a loss of 32 units resulting from A's use of his strategy A_2.

Similarly, if *B* uses his strategy B_2 he can only count on a loss of 11, the maximum of the second column which results from *A* using his strategy A_1. If *B* uses his strategy B_3 he has to consider the possibility that *A* might use in this situation his strategy A_1 so that he must count with a loss of 31 money units.

Now in our table *A* is the maximiser and *B* the minimizer. Hence *A* will try to maximize the row minima and *B* will endeavour to minimize the column maxima.

In Table 1 there is a common element which is at the same time the maximum of the row minima and the minimum of the column maxima. This is the minimax or saddle point. It is a combination of the strategies A_1 and B_2 and represents a gain of *A* (loss of *B*) of 11 money units. By using his strategy A_1, *A* can make sure that he will gain at least 11 and by using his strategy B_2, *B* can assure himself that he will not lose more than 11 units. The minimax or saddle point solves definitely the two person zero sum game, if it exists.

But since the numbers in our matrix are perfectly arbitrary the minimax or saddle point need not exist. Consider *e.g.* the following game matrix :

TABLE 2

Gain of A=Loss of B

Strategy of B

Strategy of A		B_1	B_2	B_3	row minimum
	A_1	9	10	11	9
	A_2	11	10	9	9
	A_3	12	10	8	8
Column maximum		12	10	11	

It is evident that in this case no saddle point (minimax) exists. Hence we change the problem slightly :

Assume that A and B play this game not just once but many times. Assume further that A uses his strategies A_1, A_2 and A_3 with the probabilities p_1, p_2 and p_3. Also assume that B utilizes his strategies B_1, B_2 and B_3 with the probabilities q_1, q_2 and q_3. Then both will be interested in the mathematical expectation E of gains (for A) or losses (for B) :

$$(1)\quad E=9p_1q_1+10p_1q_2+11p_1q_3+11p_2q_1+10p_2q_2+9p_2q_3+12p_3q_1 +10p_3q_2+8p_3q_3$$

Now A is looking for a maximum of the mathematical expectation and B for a minimum of the mathematical expectation E. The solutions are for A : $p_1=2/3$, $p_2=0$, $p_3=1/3$; alternatively $p_1=\frac{1}{2}$, $p_2=\frac{1}{2}$, $p_3=0$. By following either of these two mixed strategies A can make sure that in the long run he will win at least $E=10$.

B is looking for a minimum of the mathematical expectation. His solution is : $q_1=0$, $q_2=1$, $q_3=0$; alternatively he might use the following mixed strategy : $q_1=\frac{1}{2}$, $q_2=0$, $q_3=\frac{1}{2}$. If player B follows either of these stratagies he can make sure in a long series of plays to lose not more than $E=10$.

Whereas a minimax or saddle point exists not always for pure strategies in arbitrary game matrices, the minimax always exists for the mathematical expectation. The theory of von Neumann and Morgenatern (1944) was generalised by Wald (1950).

2. *Risk*

The economist understands by risk a situation in which an economic subject (individual or manager of a firm) makes a choice under the following conditions. The joint prohability distribution of the relevant factors is known. In practice this means that conditions are very stable (*e.g.* no techological progress, no change in taste) and there is very ample past experience with conditions of demand, production etc. The distinction between risk, where there are known probability distributions and uncertainty, where the probability distributions are not known, is due to Knight (1933).

Consider *e.g.* the situation of a farm in Hancock County, Ellswirth Township Iowa 1928-52. (Babbar 1955, Tintner 1955). For the sake of simplicity consider only two activities : x_1 growing corn, and x_2 growing flax. Assume the net prices of these commodities given : \$1.56 for a bushel of corn and \$3.81 for a bushel of flax. Then in the short run (neglecting fixed cost) the farmer will endeavour to maximize the linear function :

$$x = 1.56\ x_1 + 3.81\ x_2. \qquad (2)$$

This linear function is to be maximised under the conditions of production. These are represented by constant input coefficients in the short run. We consider again for simplicity only two factors of production : Land and capital. We make the assumption that labour is available in such a quantity that it imposes no restraint on production.

Using the observed mean values of the input coefficients during the period of observation the conditions of production might be represented as follows. For land :

$$0.022740\ x_1 + 0.092440\ x_2 \leqslant 148. \qquad (3)$$

The average input coefficient of land in the production of corn is 0·022740 acres. The average input coefficient of land for the production of flax is 0.092440 acres. The inequality states, that the total land used for the production of corn plus the total land utilized in the production of flax cannot be larger than the total land available on an average Iowa form, 148 acres.

The second condition refers to the utilization of capital. Using again the average values of the capital input coefficients over the whole period, we have for capital the following restrictions :

$$0.317720\ x_1 + 0.969500\ x_2 \leqslant 1800. \qquad (4)$$

It takes on the average a capital of \$0.317720 to produce a bushel of corn. Similarly, on the average it takes \$0.969500 to produce a bushel of flax. The inequality expresses the fact that the total capital available for the production of corn and flax cannot be larger than the capital at the disposal of an average Iowa form, which is $1800.

Further, we have the evident assumption that it is not possible to produce negative amounts of any commodity.

(5) $$x_1 \geqslant 0,\ x_2 \geqslant 0.$$

The solution of this maximum problem is as follows: The maximum profit is $8837.971 can be achieved, by producing $x_1=5365.266$ bushels of corn (maize) and $x_2=0$ bushel of flax. In order to achieve this result it is necessary to utilize only 128.829 acres of land, *i.e.* to leave 19.171 acres unutilised. On the other hand, all $1800 of capital will be used.

It is remarkable, that to each maximum linear programming problem there exists a dual minimum problem. In our case it is as follows. Introduce two new unknowns, the accounting or shadow prices of land and capital u_1 and u_2. These are not necessarily market prices but bookkeeping prices which enable the farmer to compute cost rationally. He will now endeavour to minimize the accounting or bookkeeping cost:

(6) $$z^*=148\ u_1+1800\ u_2.$$

The conditions under which these costs are minimized are now:

(7) $$0.022740\ u_1+0.317720\ u_2 \geqslant 1.56.$$

This condition refers to the production of corn. It states that the imputed cost of land plus the imputed cost of capital used in the production of maize must be at least as large as the net price of maize ($1·56).

The second inequality for the minimum problem is:

(8) $$0{\cdot}092440\ u_1 + 0{\cdot}969500\ u_2 \geqslant 3{\cdot}81.$$

This inequality refers to the production of flax. It states that the imputed cost of land plus the imputed cost of capital in maize production cannot be smaller than the net price of corn ($3·81).

Also, of course, the imputed or shadow prices cannot be negative:

(9) $$u_1 \geqslant 0,\quad u_2 \geqslant 0.$$

The solution of the dual minimum problem is as follows: The minimum imputed cost is $\overset{*}{z} = 8837{\cdot}971$, exactly equal to the maximum net profit z. This is achieved by making $u_1 = 0$. The imputed cost of land is zero, land is for the typical farmer a free good. This follows from the fact that he does not utilize all 148 acres available, but leaves 19·171 acres uncultivated. Land is a free good for him. It would cost him nothing to use another acre.

On the other hand, the imputed cost of capital is $u_2 = 4{\cdot}91$. Another dollar's worth of capital would be worth \$4·91 to the typical farmer.

Consider now a problem in the theory of risk. Assume that the farmer still maximizes his net profit z (2) but now under the following conditions:

(10)
$$b_{11}x_1 + b_{12}x_2 \leqslant 148$$
$$b_{21}x_1 + b_{22}x_2 \leqslant 1800$$

and also under the non-negativity conditions (5). The b_{ij} are the input coefficients for land and capital in the production of corn and flax. We assume that they are normally and independently distributed with the following arithmetic means and standard deviations. These are based upon observations during the period indicated.

TABLE 3

Input coefficient	*Arithmetic*	*Standard deviation*
b_{11}	0·022740	0·0065205
b_{12}	0 092440	0·0256583
b_{21}	0·317720	0·0853977
b_{22}	0·969500	0·4338177

By numerical methods it is possible to derive the approximate distribution of the objective function z. The arithmetic mean of net profits is \$11,081, the standard deviation 7608, skewness $\gamma_1 = 1{\cdot}095$, kurtosis $\gamma_2 = 0{\cdot}733$. The lower 95% probability limit is 2000.

The derivation of the (approximate) probability distribution of the objective function is called the passive approach in stochastic linear programming (Tintner 1960). It might be useful in comparing two farms in different states, say Iowa and Caifornia.

In the active approach to linear programming we consider on the contrary the problem fixed, instead of comparing two different problems. The policy variables in the active approach are the allocations of the available resources. Denote by u_{11} the proportion of land allocated to the growing of corn; by u_{12} the proportion of land used for the production of flax ($u_{11}+u_{12}=1$). Also, denote by u_{21} the proportion of capital used in the growing of corn; by u_{22} the proportion of capital used for the growing of flax ($u_{21}+u_{22}=1$). We also assume that the resources (land and capital) are fully utilized.

The arithmetic mean of the probability distribution of anticipated net profit z under various allocations of the resources (land and capital) is shown in the following table:

TABLE 4

Arithmetic mean of profits

Allocation of land

Allocation of capital	$u_{11}=0$ $u_{12}=1$	$u_{11}=\frac{1}{4}$ $u_{12}=\frac{3}{4}$	$u_{11}=\frac{1}{2}$ $u_{12}=\frac{1}{2}$	$u_{11}=\frac{3}{4}$ $u_{12}=\frac{1}{4}$	$u_{11}=1$ $u_{12}=0$
$u_{21}=0$ $u_{22}=1$	5704	5313	5168	2073	0
$u_{21}=\frac{1}{4}$ $u_{22}=\frac{3}{4}$	5059	6394	5981	4586	2537
$u_{21}=\frac{1}{2}$ $u_{22}=\frac{1}{2}$	4082	6027	7008	6799	4945
$u_{21}=\frac{3}{4}$ $u_{22}=\frac{1}{4}$	2539	4779	6713	7584	6925
$u_{21}=1$ $u_{22}=0$	0	2973	5075	7272	8472

The figures in this table have to be intrpreted in the following way: Assume *e.g.* that the farmer allocates one-fourth of this land to Corn ($u_{11}=\frac{1}{4}$) and hence three fourths of the land to flax ($u_{12}=\frac{3}{4}$). Assume further, that he allocates one-half of the capital for the production of corn ($u_{21}=\frac{1}{2}$) and one half of his capital to the production of flax ($u_{22}=\frac{1}{2}$). Then in the long run, on the average he can expect a net profit of \$6027, etc.

It is evident from the Table that the optimum allocation of resources is achieved, from the point of view of maximizing the mathematical expectation of profits, *i.e.* the average profit in a long series of production, is to allocate all land ($u_{11} = 1$) and all capital ($u_{21} = 1$) to the production of corn and no land ($u_{12} = 0$) and no capital ($u_{22} = 0$) to the production of flax.

The criterion of the mathematical expectation (arithmetic mean of the probability distribution) is appealing if we consider a long series of trials or experiments (Marschak 1951). But a more cautions policy would be e.g. to maximize the profit which can be achieved with *e.g.* 95% probability. This corresponds to the 95% fiducial or confidence limit well known in statistics.

TABLE 5

Lower 5% probability of profits

Allocation of land

Allocation of capital	$u_{11} = 0$ $u_{12} = 1$	$u_{11} = \frac{1}{4}$ $u_{12} = \frac{3}{4}$	$u_{11} = \frac{1}{2}$ $u_{12} = \frac{1}{2}$	$u_{11} = \frac{3}{4}$ $u_{12} = \frac{1}{4}$	$u_{11} = 1$ $u_{12} = 0$
$u_{21} = 0$ $u_{22} = 1$	3437	3199	2132	1067	0
$u_{21} = \frac{1}{4}$ $u_{22} = \frac{3}{4}$	3000	4413	3608	2660	1559
$u_{21} = \frac{1}{2}$ $u_{22} = \frac{1}{2}$	2099	3898	5133	4098	3119
$u_{21} = \frac{3}{4}$ $u_{22} = \frac{1}{4}$	1049	2806	4419	5505	4676
$u_{21} = 1$ $u_{22} = 0$	0	1465	2928	4361	5749

This table represents the profit which may be obtained with 95% probability. Consider *e.g.* the following allocation of resources: The average farmer uses half of his land for growing corn ($u_{11} = \frac{1}{2}$) and half for growing flax ($u_{12} = \frac{1}{2}$). Further, he uses three-fourths of his capital for corn production ($u_{21} = \frac{3}{4}$) and only one-fourth for the growing of flax ($u_{22} = \frac{1}{4}$). Then he might expect, with a probability of 95% a profit not smaller than $ 4419, etc.

It appears from this table that again the optimum allocation of resourcers is as follows : In order to maximize the profits which can be anticipated with a probability of 95 % the optimum allocation of resources is again : The farmer should utilize all his land for growing corn ($u_{11}=1$) and none for the production of flax ($u_{12}=0$). Also it is best to allocate all the capital to corn production ($u_{21}=1$) and no capital at all to the production of flax ($u_{22}=0$). In this way the farmer has a probability of at least 95 in hundred to secure a profit of $ 5749.

The active approach to stochastic linear programing where we consider the allocation of resources as policy variables, is more useful in practice than the passive approach, where we simply are able to compare several stochastic situations.

3. Uncertainty

Uncertainty exists if the individual in question has to act under conditions where the relevant probability distributions are not known. This situation is frequently called : Gains against nature.

It is best represented in game theoretical form. Consider an individual who plays against nature (Wald 1950). The situation is represented in the following table :

TABLE 6

Gain matrix for the individual State of nature

action of the individual	s_1	s_2	*Row minimum*
a_1	0	3*	0
a_2	1*	1	1

The interpretation of this matrix is as follows :

Suppose nature is in the state s_1. Then if the individual takes action a_1 he will gain zero; if he takes action a_2 he will gain one. Suppose on the other hand that nature is in the state s_2. Then if individual takes action a_1 he will gain three; if he takes action a_2 he will gain one.

The probabilities of the states of nature are not known. Hence Wald argues, the individual should behave like a player in a two person zero sum game. Since nature is assumed to be an intelligent opponent, the individual can only count on the row minimum, *i e.*, expects the worst for each action he takes.

We indicate the row minima in the last column of Table 6. The individual seeks the maximum of the row minima and will hence take action a_2.

The idea of treating problems of this nature as two person zero sum game assumes that nature acts like an intelligent opponent.

This is evidently not the case and the theory of Wald must be considered excessively pessimistic. Nevertheless the idea of minimax strategies must be considered one of the most important advances in statistics and probability theory comparable only to Bayes theorem and just as much subject to dispute.

Various modifications of the minimax principle have been suggested. One idea is due to Savage (1954): Minimax regret.

The individual in question considers as his regret the difference between the result of his action and the action he would have taken if he had known the state of nature. We have starred in Table 6 the maximum for each column (state of nature). Deducting these values from each value in Table 6 we derive the regret matrix.

TABLE 7

Regret matrix for the individual State of nature

Action of the individual	s_1	s_2	*Row minimum*
a_1	—1	0	—1
a_2	0	—2	—2

Again we apply the minimax principle. Nature is trying to maximize the regret of the individual. This involves the row minimum which is indicated in the last column of Table 7. The individual will choose the minimum of these negative numbers and hence action 1.

Another and older approach goes back to Bayes and Laplace: If the probabilities of the state of nature are known then equal probabilities are assigned to each state of nature. By assigning the probability $\frac{1}{2}$ to the states s_1 and s_2 we derive the following table for the mathematical expectation of each action given these *a priori* probabilities:

TABLE 8

Action of individual	*Mathematical expectation*
a_1	3/2
a_2	1

If the individual tries to maximize his mathematical expectation he will choose action a_1.

Still another approach is due to Hurwicz. Here the individual considers not only the minimum, but also the maximum which might result from his action. We present this in the following table :

TABLE 9

Action of the individual	*Row minimum*	*Row maximum*	*Hurwicz criterion*
a_1	0	3	$3\text{-}3w$
a_2	1	1	1

We introduce now, following Hurwicz, a weighted mean of the minimum and maximum, giving the minimum the weight $w(0 \leq w \leq 1)$ and the maximum the weight $1=w$. Here w might be considered a measure of pessimism for the individual. If $w=0$, he always expects the best to happen, if $w=1$ he anticipates the worst to happen in any case. This last case is of course the minimax solution discussed above.

We see in the above table that the individual will choose action a_1 if $2/3 \leq w \leq 1$, and action a_2 if $0 \leq w \leq 2/3$.

Another approach dealing with the problem of uncertainty in a situation in which the *a prori* probabilities are completely unknown is the theory of Carnap (1950 ; 1949). Consider *e.g.* a completely new commodity where no experience is available. As an example we consider tickets for a rocket ship going to the moon or Mars. Assume that there are just two customers available who are rich enough to buy tickets. According to a special theory of the Carnap (c'') the situation can be represented in the following table (Pege 12), where the numbers indicate the number of tickets bought by each customer. We assume that each of the two customers buys no, one or two tickets.

In this table we indicate the state descriptions in each line, *e.g.*, in state description 1, neighter C_1 nor C_2 buys a ticket ; in state description 2, C_1 buys no ticket but C_2 purchases 1, etc.

All state descriptions which may be obtained by permutations of the individuals (in our case C_1 and C_2) form in the Carnap theory

TABLE 10
Carnap Probability C

State description	*Number of Tickets bought*		*Probability*
	C_1	C_2	
1	0	0	1/6
2	0	1	1/12
3	1	0	1/12
4	0	2	1/12
5	2	0	1/12
6	1	1	1/6
7	1	2	1/12
8	2	1	1/12
9	2	2	1/6

a structure description. Hence state description 1 is a structure description, so are state descriptions 6 and 9, which form distinct structure descriptions.

Also, state descriptions 2 and 3 form a structure description ; so do state descriptions 4 and 5 ; also state descriptions 7 and 8 form together one structure description.

By using the Carnap the theory a^* and assigning each structure description the same *a priori* probability we obtain the probabilities in the 1st column of Table 10, if we assign the same probability to each state description within a given structure description. Calling x the number of tickets sold we obtain the following *a priori* probability distribution : (Table 11, Page 13)

Having obtained an *a priori* probability distribution we consider now the following simple inventory problem : Suppose the entrepreneur constructs a rocket with a seats ; if he actually sells x places for two money units each and it costs one unit of money to construct a place, his profit will be :

(*ii*) $P=2x-a \quad 0\leqslant x\leqslant a$

$P=a \quad x>a$

TABLE 11

A priori probability dtsrribution

Number of tickets sold	*Probability*
x	p_x
0	1/6
1	1/6
2	1/3
3	1/6
4	1/6

Now we construct the following table, which shows the situation if x tickets are sold and a rocket of a places is constructed :

TABLE 12

Profits

Size of rocket a

Number of tickets sold x	*Probability Px*	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$
0	1/6	0	—1	—2	—3	—4
1	1/6	0	1*	0	—1	—2
2	1/3	0	1	2*	1	0
3	1/6	0	1	2	3	2
4	1/6	0	1	2	3	4
Mathematical expectation		0	2/3	1*	2/3	0

Consider *e.g.* the case where a rocket of size $a=2$ is constructed The cost is 2 units. If no ticket is sold ($x=0$) the profit is -2. If $x=1$ unit is sold the profit is 0. In case $x=2$ places are sold, the profit is 2 units, and the same profit is realized for any $x\geqslant 2$.

Which decision criterion should be adopted ? Consider first the mathematical expectation. This is computed with the help of *a priori* probabilities exhibited in the second column of the table. If this criterion is adopted, then a rocket of size $a=2$ should be constructed.

But instead of the mathematical expectation of the probability distribution of profits we might also consider the mode (largest value) or median (middle value) of the distribution. The mode occurs with $x=2$ and this is also the median, since the distribution is symmetrical. If the modal or median profit is considered, a rocket of size $a=2$ should be constructed.

Finally, consider the highest profit which can be made with probability 2/5. This corresponds to the lower 33% confidence or fiducial limit in statistics. If this point of view is adopted this choice corresponds to $x=1$ and it follows that a rocket of size $a=1$ should be constructed.

This is certainly a possible approach with no previous information at all. The situation is slightly different if we have a small sample available. As the sample becomes large we have the problem previously discussed under the heading of risk.

Consider now a situation where a (small) sample of s persons is available. Of these s persons s_0 have bought no ticket, s_1 one ticket, s_2 two tickets.

We want to make a prediction for a new sample of size s'. The probability that there will be s'_0 people, who buy no tickets, s'_1 who buy one and s'_2 who purchase two tickets, follows from Carnap's (1950) theory of predictive inference.

Now we make the following assumptions : We have available a past sample of $s=3$ persons ; of these $s_0=2$ have not bought any tickets, $s_1=1$ has bought one place and $s_2=0$ have bought two tickets.

We predict for a new sample of $s'=3$ persons. Denoting by s'_0, s'_1 and s'_2 the number of people who will buy zero, one and two tickets, we derive for the probability the following formula for Carnap's predictive inference :

$$p=\frac{\binom{2+s'_0}{s'_0}\binom{1+s'_1}{s'_1}\binom{s'_2}{s'_2}}{56} \tag{12}$$

This is the probability that we obtain in a sample of $s'=3$ customers the following results: s'_0 buy no ticket, s'_1 purchase one ticket, s'_2 buy two tickets. The s_0', s'_1, s'_2 are non-negative numbers and

$$s'_0+s'_1+s'_2=3. \tag{13}$$

Making exactly the same assumptions as below we represent the situation in the following table :

TABLE 13

Profits

Size of the ticket a

Number of tickets sold x	*probability* p_x	$a=0$	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
0	10/56	0	−1	−2	−3	−4	−5	−6
1	12/56	0	1*	0	−1	−2	−3	−4
2	15/56	0	1	2*	1	0	−1	−2
3	10/56	0	1	2	3	2	1	0
4	6/56	0	1	2	3	4	5	2
5	2/56	0	1	2	3	4	5	4
6	1/56	0	1	2	3	4	5	6
mathematical expectation		0	46/56	48/56*	30/56	−8/56	−58/56	−112/56

Now if the manager bases himself upon the mathematical expectation computed with the help of Carnap's theory of predictive inference, he will construct a rocket with $a=2$ places. If he bases himself upon the mode of the probability distribution, he notes that the mode occurs with $x=2$. Hence again from the point of view he will construct a rocket with $a=2$ places.

Suppose however he wants to maximize the profit which can be made with at least 41/56=76% probability. This corresponds to $x=1$ and would induce him to construct a rocket only$a=1$ places.

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